AN EXAMPLE OF A UNIVERSAL BANACH SPACE

BY

A. SZANKOWSKI

ABSTRACT

We construct a separable reflexive Banach space $\mathfrak{X}$ which is complementably universal for all finite dimensional Banach spaces. By this we mean: for every finite dimensional Banach space E there is isometric embedding i: $E \longrightarrow \mathfrak{X}$ such that there exists a projection $P: \mathfrak{X} \xrightarrow{\text{onto}} iE$ with $\| P \| = 1$.

Our result is related to that of C. Bessaga [1] who proved that there exists no finite dimensional B-space universal for all two dimensional B-spaces. Bessaga's result was extended by V. Klee in [2] and by J. Lindenstrauss in [3]. In these papers the problem of the existence of a separable reflexive space universal for all finite dimensional spaces was posed and discussed. It was noticed there that the space $A = (\Sigma \oplus l^n_\infty)_2$ is "almost" universal for all finite dimensional spaces (by this we mean that for every $\varepsilon > 0$ and for every finite dimensional E there exists an embedding $T: E \to A$ such that $\| T \| \| T^{-1} \| < 1 + \varepsilon$). In our construction we develop in some sense this idea. Let us mention also that J. Lindenstrauss proved in [4] that L_1 is a universal space for all two dimensional spaces.

The existence of $\mathfrak{X}$ will immediately follow from the following:

THEOREM. *For every natural m there exists a Banach space $\mathfrak{X}_m$ which is isomorphic to l_2 and is complementably universal for all m-dimensional Banach spaces. (We take $\mathfrak{X} = (\Sigma \oplus \mathfrak{X}_m)_2$.)*

PROOF. m will be fixed. By a *gauge function* we mean any continuous function $w: R^m \to R_+$ (non-negative reals) such that

1) $w(tx) = |t| w(x)$ for every $t \in R$, $x \in R^m$.

2) $w(x) = 0$ iff $x = 0$.

We denote $K(w) = \{x \in R^m : w(x) \leqq 1\}$, $\mathfrak{A} = \{K(w): w$ is a gauge function$\}$.

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$\mathfrak{A}$ will also be regarded as the set of all gauge functions and for $K \in \mathfrak{A}$, by $w(K)$ we denote the "Minkowski functional" of K.

For $K \in \mathfrak{A}$ we define its deficiency

$$D(K) = \inf\{t: tK \supset \operatorname{conv} K\}.$$

Let $\mathfrak{B} = \{K \in \mathfrak{A}: D(K) < 2\}$. It is clear that $\mathfrak{B}$ is separable in the following sense:

(S) There exists a sequence $\{w_i\}_{i=1}^{\infty} \subset \mathfrak{B}$ such that for every pair $w, w' \in \mathfrak{B}$ with $w < w'$ there is an i such that

$$w < w_i < w'.$$

In the sequel we use the following notation:

1) Given $w \in \mathfrak{B}$, let $\bar{w}$ be the norm corresponding to the set conv $K(w)$. By the definition of $\mathfrak{B}$,

$$\bar{w} \leqq w \leqq 2\bar{w}. \tag{1}$$

2) For a sequence $\boldsymbol{x} = (x_i) \in (R^m)^\infty$, let $W(\boldsymbol{x}) = (\sum w_i^2(x_i))^{\frac{1}{2}}$, $\bar{W}(\boldsymbol{x}) = (\sum \bar{w}_i^2(x_i))^{\frac{1}{2}}$. By (1),

$$\bar{W} \leqq W \leqq 2\bar{W}. \tag{1'}$$

Denote $X = \{\boldsymbol{x}: \bar{W}(\boldsymbol{x}) < \infty\}$, $B = \{\boldsymbol{x}: \bar{W}(\boldsymbol{x}) \leqq 1\}$. Then the Banach space $\mathfrak{X}' = X$ equipped with the norm $\bar{W}$, is an l_2 product of a countable family of m-dimensional spaces and is therefore isomorphic to l_2.

3) For $x \in R^m$ and a sequence $M = (i_1, i_2, \cdots)$ of natural numbers put

$$(w_{i_1} \overset{2}{\vee} \cdots \overset{2}{\vee} w_{i_n})(x) = \left(\sum_{j=1}^{n} w_{i_j}^2(x)\right)^{\frac{1}{2}},$$

$$(\overset{2}{\vee} w_i)_{i \in M}(x) = \left(\sum_{j=1}^{\infty} w_{i_j}^2(x)\right)^{\frac{1}{2}}.$$

In other words,

$$(\overset{2}{\vee} w_i)_{i \in M}(x) = W(x_M)$$

where

$$x_M = (x_i) \text{ with } x_i = x \text{ for } i \in M,\ x_i = 0 \text{ for } i \notin M.$$

Denote

$$\Delta_M = \{x_M : x \in R^m\} \text{ and } B_M = \{\boldsymbol{x} \in \Delta_M : W(\boldsymbol{x}) \leqq 1\}.$$

We fix once and forever a norm u in R^m.

LEMMA. *Let v be a norm in R^m and let (a_n) be a sequence of positive numbers such that*

(i) $$\sum_{n=1}^{\infty} a_n^2 = 1, \quad a_1^2 > \tfrac{1}{2} > a_n^2 \text{ for } n = 2,3,\cdots.$$

Then there exists a sequence $M = M(v) = (i_n)$ such that

(*) $$v = (\overset{2}{\vee} w_i)_{i \in M}$$

(**) $$\tfrac{1}{2} a_n v \leqq w_{i_n} \leqq 2a_n v, \text{ more precisely}$$

$$a_1 v \leqq w_{i_1} \leqq 2a_1 v, \ \tfrac{1}{2}a_n v \leqq w_{i_n} \leqq a_n v \text{ for } n = 2,3,\cdots.$$

PROOF OF THE LEMMA. By induction we shall find (i_n) so that

(2) $$A_n v < W_n < A_n(1 + a_{n+1}^2/2A_n^2)^{\frac{1}{2}} v,$$

where

$$A_n = \left(\sum_{j=1}^{n} a_j^2\right)^{\frac{1}{2}}, \ W_n = w_1 \overset{2}{\vee} \cdots \vee w_{i_n}.$$

By (S) we can find a suitable i_1. Denote for $n = 1,2,\cdots$

$$s_n = (A_{n+1}^2 v^2 - W_n^2)^{\frac{1}{2}}.$$

By (2) we have

(3) $$\tfrac{1}{2}a_{n+1}v < s_n < a_{n+1}v.$$

Hence $D(s_n) < 2$ and $s_n \in \mathfrak{B}$. Therefore, by (S), there is a number i_{n+1} such that

(4) $$s_n < w_{i_{n+1}} < (1 + \alpha)^{\frac{1}{2}} s_n,$$

where α is any number satisfying

(5) $$(1 + \alpha)^{\frac{1}{2}} s_n < a_{n+1} v$$

(5′) $$\alpha < a_{n+2}^2.$$

Now if we estimate $W_{n+1}^2 = W_n^2 + w_{i_{n+1}}^2$:

$$\begin{aligned} A_{n+1}^2 v^2 &= W_n^2 + s_n^2 < W_n^2 + w_{i_{n+1}}^2 < W_n^2 + s_n^2 + \alpha s_n^2 \\ &\leqq (A_{n+1}^2 + \alpha a_{n+1}^2) v^2 \leqq A_{n+1}^2 (1 + a_{n+2}^2/2A_{n+1}^2) v^2, \end{aligned}$$

then we get (2).

It is clear that (i) and (2) imply (*). Also (i), (2) imply (**) for $n = 1$ and (3), (4), (5) imply (**) for $n > 1$.

CONSTRUCTION. To each norm $v > u$ in R^m we assign a (fixed) sequence (a_n) so that (i) and (ii), (iii), (iv) below are satisfied:

(ii) $$\sum_{n=2}^{\infty} a_n^2 \leqq 2^{-12}$$

(iii) $$a_1 v > u > a_n v \text{ for } n = 2, 3, \cdots,$$

(iv) $$\sum_{j=k+1}^{\infty} a_j^2 \leqq 2a_k^2, \text{ i.e., } \sum_{j=k}^{\infty} a_j^2 \leqq 2a_k^2 \text{ for } k = 1, 2, \cdots.$$

Then, by the Lemma, we can fix a sequence $M(v) = (i_n)$ so that (*), (**) are satisfied.

Now we define $\mathfrak{X}_m$ as X equipped with the norm induced by the convex set

$$K = \overline{\text{conv}} \left\{ 2^{-5}B \vee \bigcup_{v>u} B_{M(v)} \right\}$$

(the closure is taken in the sense of W (or $\bar{W}$)).

Since $2^{-5}B \subset K \subset B$, it is clear that the identity map is an isomorphism from $\mathfrak{X}'$ onto $\mathfrak{X}_m$. Therefore $\mathfrak{X}_m$ is also isomorphic to l_2.

The theorem will be proved if we check the following two facts (here $M = M(v) = (i_n)$ for a (fixed) norm $v > u$; (a_n) is the sequence of numbers assigned to v):

1) $K \cap \Delta_M = B_M$ (then the map $x \to x_M$ is an isometric embedding of R^m equipped with the norm v into $\mathfrak{X}_m$; Δ_M is the range of this embedding),

2) There exists a projection $P: X \xrightarrow{onto} \Delta_M$ such that $P(K) \subset B_M$, i.e.,

a) $P(2^{-5}B) \subset B_M$

b) $P(B_{M'}) \subset B_M$ for any $M' = M(s) = (q_n)$ with $s > u$.

One sees easily that 2) implies 1).

For $\boldsymbol{x} = (x_i)$ we define $P(\boldsymbol{x}) = x_M$, where

$$x = A \cdot \sum_{j=1}^{\infty} b_j x_i ,$$

where $b_1 = a_1^2$, $b_n = 8a_n^2$ for $n = 2, 3, \cdots$ and $A = (\sum_{j=1}^{\infty} b_j)^{-1}$.

By (i) and (ii) we have $1 \geqq A \geqq \frac{1}{2}$.

PROOF OF a).

$$\bar{W}(Px) = v(x) \leqq 8A \sum_{j=1}^{\infty} a_j^2 v(x_{i_j}) \leqq 8(\Sigma a_j^2)^{\frac{1}{2}} \cdot (\Sigma a_j^2 v^2 (x_{i_j}))^{\frac{1}{2}}$$

$$\leqq 8 \cdot 1 \cdot 2(\Sigma w_{i_j}^2(x_{i_j}))^{\frac{1}{2}} \leqq 2^4 W(x) \leqq 2^5 \bar{W}(x).$$

PROOF OF b). We see that for $x \in R^m$, $P(x_{M'}) = P(x_{M \cap M'}) = cx_M$ where $c = A \cdot \sum_{j=1 j \in M} b_j$.

We consider two cases:

(i) $q_1 = i_1$. Let k be the first number such that $i_k \notin M'$. Then $c^2 \leqq c \leqq 1 - 8Aa_k^2 \leqq 1 - 4a_k^2$ and

$$\begin{aligned} W^2(Px_{M'}) &= c^2 \sum_{j=1}^{\infty} w_{i_j}^2(x) \\ &\leqq (1 - 4a_k^2) w_{i_1}^2(x) + \sum_{1<j<k} w_i^2(x) + \sum_{j \geqq k} w_i^2(x) \\ &\leqq \sum_{1 \leqq j < k} w_{i_j}^2(x) - (4a_k^2 w_{i_1}^2(x) - \sum_{j \geqq k} w_{ij}^2(x)) \\ &\leqq W^2(x_{M'}). \end{aligned}$$

(We have by (iv) and (**): $4a_k^2 w_{i_1}^2(x) \geqq \sum_{j \geqq k} w_{ij}^2(x)$.)

(ii) $q_1 \neq i_1$. Then, by (iii), $q_1 \notin M$ and

$$\begin{aligned} \bar{W}(x_{M \cap M'}) &\leqq W(x_{M \cap M'}) \leqq \left(\sum_{j=2} w_{q_j}(x) \right)^{\frac{1}{2}} \\ &\leqq 2^{-6} s(x) = 2^{-6} W(x_{M'}) \leqq 2^{-5} \bar{W}(x_M). \end{aligned}$$

Therefore, if $x_{M'} \in B_{M'}$, then $x_{M \cap M'} \in 2^{-5}B$, and, by a), $P(x_{M \cap M'}) \in B_M$, hence $P(x_{M'}) = P(x_{M \cap M'}) \in B_M$.

This completes the proof of our theorem.

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REFERENCES

1. C. Bessaga, *A note on universal Banach spaces of finite dimension*, Bull. Acad. Polon. Sci. **6** (1958), 97–101.

2. V. Klee, *Polyhedral sections of convex bodies*, Acta. Math. **103**. (1960), 243–267

3. J. Lindenstrauss, *Notes on Klee's paper 'Polyhedral sections of convex bodies'*, Israel J. Math. **4** (1966), 235–242.

4. J. Lindenstrauss, *On the extension of operators with a finite dimensional range*, Illinois J. Math. **8** (1964), 488–499.

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